COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 11 (3) (1991) 295–297

NOTE

A NOTE ON THE SHARP CONCENTRATION OF THE CHROMATIC NUMBER OF RANDOM GRAPHS

TOMASZ ŁUCZAK

Received February 2, 1989

Let $G(n,p)$ be a random graph on n vertices in which each possible edge is present independently with probability $p = p(n)$. We say that the chromatic number of $G(n,p)$ is *concentrated in width s for $p = p(n)$* if for some function $u = u(n)$ we have

$$\lim_{n\to\infty} \text{Prob}(u \le \chi(G(n,p)) \le u + s) = 1 .$$

The question of the width of concentration of the chromatic number $\chi(G(n,p))$ was first considered by Shamir and Spencer in [3] who, among others, proved also the following result on the concentration of the chromatic number in sparse random graphs.

Theorem 1. *If $p(n) < n^{-5/6-\epsilon}$, $\epsilon > 0$, then the chromatic number of $G(n,p)$ is concentrated in width 3.*

In this note we improve original argument of Shamir and Spencer and show that in the Theorem above 4 can be replaced by, clearly the best possible, 1. Thus our main result goes as follows.

Theorem 2. *If $p(n) < n^{-5/6-\epsilon}$, $\epsilon > 0$, then the chromatic number of $G(n,p)$ is a.s. two point concentrated.*

The proof of Theorem 1 in [3] is based on two facts. The first one, the "meat" of the paper, is shown using martingales technique whereas the second follows easily from the first moment method.

Fact 1. *Let $k = k(n)$ be such that*

$$(*) \qquad \text{Prob}(\chi(G(n,p)) \ge k) > \frac{1}{\log\log n} .$$

Then a.s. all but at most $\log^2 n\,\sqrt{n}$ vertices of $G(n,p)$ can be properly coloured using k colours.

Fact 2. *Let $0 < \epsilon < 0.1$ and $d(n) = np(n)$. Then a.s. each subgraph H of $G(n,p)$ on less than $nd^{-3(1+2\epsilon)}$ vertices has less than $(1.5 - \epsilon)|H|$ edges.*

AMS subject classification (1980): 05 C 80

Remark. Actually, Fact 1 as stated above does not appear in the paper of Shamir and Spencer who use an analogous but more complicated result. Here we follow a suggestion of Alan Frieze who ingeniously noticed that Fact 1 is an immediate consequence of Shamir and Spencer's concentration theorem.

One can easily see that the assertion of Theorem 1 follows from Facts 1 and 2. Indeed, from Fact 1 a.s. $n - \log^2 n\sqrt{n}$ vertices can be coloured with k colours and Fact 2 implies that each subgraph H with all vertices uncoloured has the minimal degree less than three, so, all uncoloured vertices can be coloured using at most three additional colours (see, for example, Bollobás [1] Theorem V.1).

We shall show how to use the fact that $G(n,p)$ is very sparse more efficiently and colour all vertices of $G(n,p)$ using only one extra colour.

Proof of Theorem 2. If $np(n) \to 0$ then a.s. $G(n,p)$ consists of isolated trees so $\chi(G(n,p)) = 1$ or 2. For $1/\log n < np(n) < 1.0001$ $G(n,p)$ a.s. contains at least one edge and no subgraphs with minimal degree larger then 2 (see Łuczak [2]) so $\chi(G(n,p)) = 2$ or 3. Thus we may assume that $np(n) > 1.0001$ and, since then $G(n,p)$ a.s. contains an odd cycle, $\chi(G(n,p)) \geq 3$.

Let $1.0001 \leq d(n) = np(n) < n^{1/6-\epsilon}$ and $k = k(n) \geq 3$ be a number for which $(*)$ holds. From Fact 1 it follows that a.s. all vertices of $G(n,p)$ outside some set S, $|S| < s_0 = \log^2 n\sqrt{n}$, can be coloured by k colours. We shall prove that a.s. $\chi(G(n,p)) \leq k+1$.

Construct an increasing sequence $\{U_i\}_{i=0}^{m}$ of subsets of $G(n,p)$ setting $U_0 = S$ and for given $U_0, U_1, \ldots, U_i$ define U_{i+1} as $U_i \cup \{w_1, w_2\}$ where $w_1, w_2 \notin U_i$ are joined by an edge and both of them are adjacent to some vertices of U_i. If such pair w_1, w_2 does not exist finish construction with $m = i$.

Now note that $m \leq m_0 = \log^3 n\sqrt{n}$. Indeed, otherwise a subgraph induced by U_{m_0} would have

$$\bar{s} = s_0 + 2m_0 \leq \log^4 n\sqrt{n} < nd^{-3(1+2\epsilon)}$$

vertices and at least $3m_0 \geq (1.5-\epsilon)\bar{s}$ edges contradicting Fact 2.

Hence there exists a set U_m, $U_m \supseteq S$ such that

$$|U_m| \leq m_0 + s_0 < nd^{-3(1+2\epsilon)}$$

and the set $N(U_s)$ of all neighbours of U_m is independent. Thus colour all vertices outside $U_m \cup N(U_m)$ using k colours, $N(U_m)$ with colour $k+1$ and U_m using colours 1, 2 and 3 (since U_m is small so, by Fact 2, the subgraph induced by it in $G(n,p)$ contains no subgraphs H with $\delta(H) \geq 3$ and thus is 3-colourable). ∎

References

[1] B. BOLLOBÁS: "Graph Theory — An Introductory Course", Springer-Verlag, 1979.

[2] T. ŁUCZAK: The size and connectivity of the k-core of a random graph *Discrete Math*, to appear.

[3] E. SHAMIR, AND J. SPENCER: Sharp concentration of the chromatic number on random graphs $G_{n,p}$ *Combinatorica* **7** (1987), 124–129.

Tomasz Łuczak

Department of Discrete Mathematics
Adam Mickiewicz University
Poznań, Poland
tomasz@plpuam11.bitnet